Kinetic roughening, global quantities, and fluctuation-dissipation relations

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Abstract. Growth processes and interface fluctuations can be studied through the properties of global quantities. We here discuss a global quantity that not only captures better the roughness of an interface than the widely studied surface width, but that is also directly conjugate to an experimentally accessible parameter, thereby allowing us to study in a consistent way the global response of the system to a global change of external conditions. Exploiting the full analyticity of the linear Edwards-Wilkinson and Mullins-Herring equations, we study in detail various two-time functions related to that quantity. This quantity fulfills the fluctuation-dissipation theorem when considering steady-state equilibrium fluctuations.

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1. Introduction

Due to its omnipresence in many fields in physics and engineering, kinetic roughening has attracted much attention over the years, see [1, 2, 3, 4, 5] for some earlier reviews. In many instances the properties of growth processes can be understood by analysing rather simple Langevin equations. These Langevin equations, which can be either linear or non-linear, provide the theorists with a class of systems that they can study in a systematic way, using a variety of techniques. In addition, these Langevin equations have a wide range of applications, ranging from different growth processes to equilibrium step fluctuations at the surface of a crystal [6, 7, 8, 9, 10] and kinetic smoothening of interfaces [11].

The simplest Langevin equations studied in this context are the linear Edwards-Wilkinson (EW) [12]

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \nu \nabla^2 h(\mathbf{x},t) + \eta(\mathbf{x},t) \tag{1}$$

and noisy Mullins-Herring (MH) [13] equations

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = -\nu \nabla^4 h(\mathbf{x},t) + \eta(\mathbf{x},t), \tag{2}$$

where the noise is usually assumed to have zero mean and to be uncorrelated:

$$\langle \eta(\mathbf{x},t) \rangle = 0 \qquad ; \qquad \langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t') \rangle = D\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t') \ . \tag{3}$$

Here $\mathbf{x}$ is a d-dimensional position vector located on the surface of the substrate, whereas for growth processes $h(\mathbf{x},t)$ is the height of a column at position $\mathbf{x}$ at time t.‡

These equations, which depend on the two parameters ν (for the EW equation ν is the surface tension or elastic constant) and D (the noise strength), are used to describe deposition with different relaxation mechanisms. For the EW equation the surface current responsible for the smoothening can be viewed to be due to the gravitational potential, whereas for the MH equation a surface current arises because of a chemical potential difference. A standard way to analyse kinetic roughening is to study the surface width, which typically displays the following three regimes. At the early stages the surface grows in an uncorrelated way during a regime that is sometimes called the random deposition regime. At a first crossover time t_1, correlated growth sets in. This correlated regime continues until a second crossover time t_2 at which the system enters the steady state regime. This time t_2 depends on the size of the system and shifts to larger values when increasing the linear extend of the substrate. For an infinitely large substrate t_2 diverges and the system never reaches the steady state.

Quite some attention was paid recently to the ageing processes [14] that take place during the correlated growth regime [15, 16, 17, 18, 19, 20, 21, 22, 23]. Most of the studies focused on local quantities as for example the height-height correlation function,

‡ In the following we mostly use the language of growth processes, but due to the different physical situations described by the same Langevin equations our results have a broader range of applications.

the response of the height to a local perturbation, the two-time roughness or the two-time incoherent scattering function [15, 16, 17, 18, 19, 20]. In [21] we discussed the correlation function of the squared width and the response of the squared width to a global perturbation. Studying changes both in ν and D, our work revealed that the global response of the surface width depends on how the system is perturbed. In addition, we showed that in the correlated regime the limit value of the corresponding fluctuation-dissipation ratio only yields the trivial value zero, due to the fact that the studied quantity, the square of the surface width, is not conjugate to any of the system parameters that are changed in our protocols.

In this paper we propose to study a global quantity directly related to the effective Hamiltonian (m being an even number)

$$H_m = \frac{\nu}{2} \int d^d x \left(\nabla^{m/2} h \right)^2 \tag{4}$$

showing up in the Hamiltonian description that yields the corresponding stochastic equation of motion:

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = -\frac{\delta H_m}{\delta h(\mathbf{x},t)} + \eta(\mathbf{x},t). \tag{5}$$

Inserting (4) into (5) yields the linear Langevin equation

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = -\nu (i\nabla)^m h(\mathbf{x},t) + \eta(\mathbf{x},t), \tag{6}$$

where for $m = 2$ we recover the EW equation, whereas $m = 4$ yields the MH equation.

As we discuss in the following, our quantity, which can be computed exactly for linear Langevin equations, has many advantages. On the one hand, it better describes the roughness of a surface than the surface width itself. On the other hand, as this quantity is conjugate to ν, a change in ν yields a global response that allows us to study the corresponding fluctuation-dissipation ratio [24, 25]. For the case of an equilibrium steady state (as it is for example encountered for step fluctuations on crystal surfaces) the celebrated fluctuation-dissipation theorem is recovered, something that was not the case for the surface width [21].

The paper is organized in the following way. In Section 2 we introduce our quantity that we calculate exactly for the linear Langevin equations used in the context of non-equilibrum growth and related problems. Section 3 is devoted to the corresponding correlation and response functions. We thereby show that we recover the fluctuation-dissipation ratio for equilibrium steady states. Section 4 gives our conclusions.

2. Global quantity conjugate to ν

Inspection of the effective Hamiltonian (4) allows us to define both for the Edwards-Wilkinson and the Mullins-Herring equations the following time-dependent global quantity that is conjugate to ν:

$$G_m(t) = \frac{1}{2} \int d^d x \left(\nabla^{m/2} h \right)^2 \ , \tag{7}$$

For $m = 2$ the quantity G_m is of course readily identified with the total "kinetic energy." In the following we consider as substrate d-dimensional lattices with linear extend L. For $m = 2$ our quantity can then be written as

$$G_2(t) = \frac{1}{2} \sum_{\mathbf{x}} \sum_{i=1}^{d} \Big[h(\mathbf{x} + \mathbf{a}_i, t) - h(\mathbf{x}, t) \Big]^2 \tag{8}$$

whereas for $m = 4$ we obtain:

$$G_4 = \frac{1}{2} \sum_{\mathbf{x}} \left\{ \sum_{i=1}^{d} \Big[h(\mathbf{x} + \mathbf{a}_i, t) - 2h(\mathbf{x}, t) + h(\mathbf{x} - \mathbf{a}_i, t) \Big] \right\}^2 . \tag{9}$$

The vector $\mathbf{x}$ now labels all points of the substrate lattice, whereas the vectors $\mathbf{a}_i$, $i = 1, \cdots, d$, indicate the primitive vectors on the substrate. For simplicity the lattice constant is assumed to be unity. Note that geometrically G_2 is the total surface slope § and G_4 is the sum of the curvatures.

Usually the roughness of a growing interface of columns of hight $h(\mathbf{x})$ at substrate site $\mathbf{x}$ is expressed by the surface width

$$W(t) = \sqrt{\frac{1}{L^d} \sum_{\mathbf{x}} \Big(h(\mathbf{x}, t) - \overline{h}(t) \Big)^2} \tag{10}$$

where $\overline{h}(t) = \frac{1}{L^d} \sum_{\mathbf{x}} h(\mathbf{x}, t)$ is the average height at time t. The quantity G_m can also be used to quantify the roughness of a surface ‖. In fact, it can even be argued that G_m captures the roughness much better than W. To see this, consider the two surfaces shown in Fig. 1 which have the same number of deposited particles. Whereas intuitively we would judge the surface (b) to be rougher than the surface (a), the surface width yields the same value for both cases. The value of G_m, on the other hand is larger for surface (b) than for surface (a), and this for both cases $m = 2$ and $m = 4$.

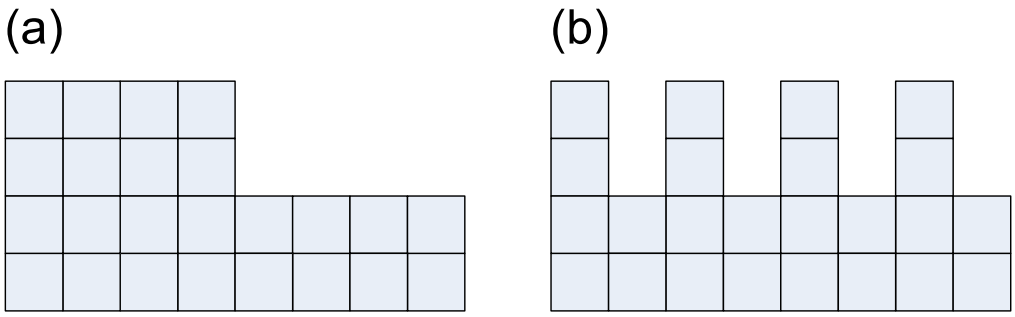

Figure 1. Two surfaces with the same surface width but different values of G_m.

For our purpose we need to derive the exact expressions for the average of G_m from the solution of the corresponding linear Langevin equations. Writing both the

§ A generalisation of this quantity to the height difference between columns separated by a displacement $\mathbf{r}$ has been called the second order height difference correlation function [4].

‖ See [26] for an example where the surface slope G_2 has been analysed in a numerical study

height $h(\mathbf{x},t)$ and the noise $\eta(\mathbf{x},t)$ as a sum over reciprocal lattice vectors, we obtain the following expression

$$\langle G_m\rangle = 2^{\frac{m-2}{2}} L^d \sum_{\mathbf{q}\neq 0} \left\langle h_{\mathbf{q}} h_{-\mathbf{q}}\right\rangle \mathcal{P}(\mathbf{q})^{m/2}, \tag{11}$$

where

$$\mathcal{P}(\mathbf{q}) = \sum_{i=1}^{d} \Big(1-\cos(q_i)\Big). \tag{12}$$

Here $\langle\cdots\rangle$ indicates an average over the noise. The averaged quantity $\langle G_m\rangle$ is therefore a sum over two-point correlation functions weighted by the factor $\mathcal{P}(\mathbf{q})^{m/2}$. Using the expression [21]

$$\langle h_{\mathbf{q}}(t_1) h_{\mathbf{p}}(t_2)\rangle = \frac{D}{L^d\nu} e^{-\nu(q^m t_1 + p^m t_2)} \frac{1}{q^m+p^m}\left(e^{\nu(q^m+p^m)t_<}-1\right)\delta^d_{\mathbf{q}+\mathbf{p}} \tag{13}$$

for the two-point correlation function, with $q=|\mathbf{q}|$, $p=|\mathbf{p}|$, and $t_<$ being the smaller of the times t_1 and t_2, we obtain

$$\langle G_m\rangle = 2^{\frac{m-4}{2}} \frac{D}{\nu} \sum_{\mathbf{q}\neq 0} \frac{1}{q^m}\left(1-e^{-2q^m\nu t}\right)\mathcal{P}(\mathbf{q})^{m/2}. \tag{14}$$

The behaviour of $\langle G_m\rangle$ is therefore controlled by the length scale $l_t \equiv (2\nu t)^{1/m}$, similar to the surface width [21]. Depending on the relation of l_t to the maximum and minimum values of q, $q_{max}=\pi\sqrt{d}$ and $q_{min}=2\pi/L$, different regimes can be discussed.

1. For $l_t < 1/q_{max}$, the system is in the random deposition regime. An expansion in small l_t yields the expression

$$\begin{aligned}\langle G_m\rangle &\approx 2^{\frac{m-2}{2}} Dt \sum_{\mathbf{q}\neq 0} \mathcal{P}(\mathbf{q})^{m/2} \\ &\approx 2^{\frac{m-2}{2}} Dt \left(\frac{L}{\pi}\right)^d \int_0^\pi d\mathbf{q}\mathcal{P}(\mathbf{q})^{m/2}\end{aligned} \tag{15}$$

such that

$$\langle G_2\rangle \approx dL^d Dt, \tag{16}$$

and

$$\langle G_4\rangle \approx d(2d+1)L^d Dt. \tag{17}$$

In this regime $\langle G_m\rangle$ varies linearly in time and shows the same time dependence as the squared surface width [21].

2. When $l_t > 1/q_{min}$, the system is in the saturation regime and $\langle G_m\rangle$ reaches its maximal value,

$$\begin{aligned}\langle G_m\rangle = G_{s,m} &\approx 2^{\frac{m-4}{2}} \frac{D}{\nu} \sum_{\mathbf{q}\neq 0} \frac{1}{q^m}\mathcal{P}(\mathbf{q})^{m/2} \\ &\approx 2^{\frac{m-4}{2}} \frac{D}{\nu}\left(\frac{L}{\pi}\right)^d I(m,d),\end{aligned} \tag{18}$$

with $I(m,d) \equiv \int_0^\pi \mathcal{P}(\mathbf{q})^{m/2}/q^m d\mathbf{q}$. The numerically evaluated values of $I(m,d)$ for various d and m are shown in Table (1).

	d=1	d=2	d=3	d=4
m=2	1.21532	3.43997	10.29687	31.53744
m=4	0.49733	1.26083	3.55376	10.51476

Table 1. The numerically evaluated values of $I(m,d) \equiv \int_0^\pi \mathcal{P}(\mathbf{q})^{m/2}/q^m d\mathbf{q}$

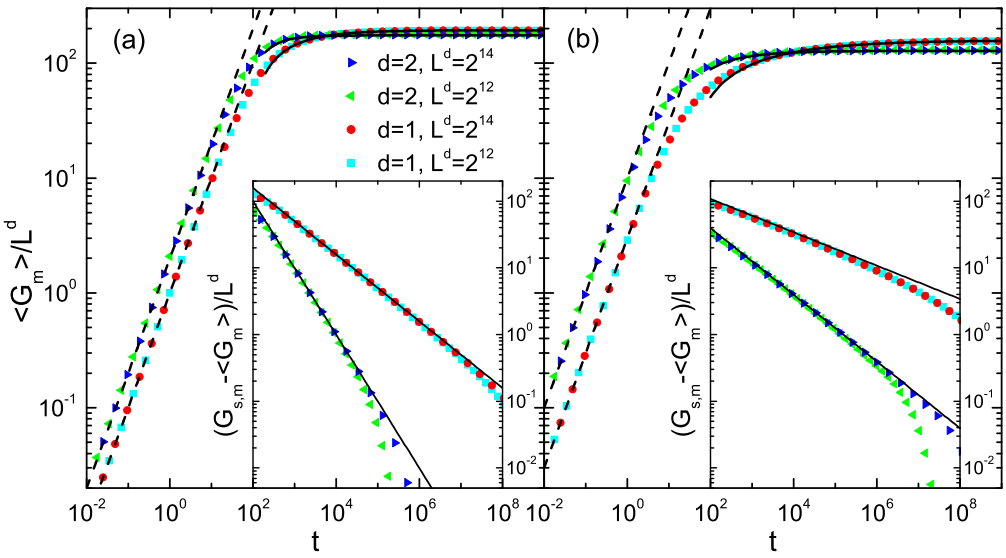

Figure 2. $\langle G_m \rangle$ for (a) the Edwards-Wilkinson case $m = 2$ and (b) the Mullins-Herring case $m = 4$. Systems of various lengths and various dimensionality of the substrate are shown. The insets display the power-law approach to the stationary value $G_{s,m}$. The symbols are obtained by numerically evaluating the exact expression Eq. (14). The dashed and solid lines are given by the asymptotic expressions (16,17) and (20). The system parameters are $\nu = 0.001$ and $D = 1$.

3. To calculate the asymptotic behavior in the correlated regime, where $1/q_{max} < l_t < 1/q_{min}$, we rewrite Eq. (14) in the form

$$\langle G_m \rangle = G_{s,m} - 2^{\frac{m-4}{2}} \frac{D}{\nu} \sum_{\mathbf{q}\neq 0} \frac{1}{q^m} e^{-2q^m \nu t} \mathcal{P}(\mathbf{q})^{m/2} \tag{19}$$

and take advantage of the hyperspherical symmetry of $\mathcal{P}(\mathbf{q})$ for $q < 1$: $\mathcal{P}(\mathbf{q}) \approx q^2/2$. In fact, since (19) involves the factor $e^{-q^m l_t^m}$, we can neglect contributions with $q > 1$, provided that $l_t \gg 1$. In that case we can simply replace the sum by an integral, yielding the result

$$\begin{aligned} \langle G_m \rangle &\approx G_{s,m} - \frac{D}{4\nu} \sum_{\mathbf{q}\neq 0} e^{-2q^m \nu t} \\ &\approx G_{s,m} - \frac{D}{4\nu} \left(\frac{L}{2\pi} \right)^d \Omega_d \frac{\Gamma(\frac{d}{m})}{m} (2\nu t)^{-\frac{d}{m}} \end{aligned} \tag{20}$$

for the correlated regime, where Ω_d is the surface of the $d-$dimensional unit sphere.

Figure 2 shows $\langle G_m \rangle$, obtained by numerically evaluating Eq. (14), for various values of m and d and various system sizes. Also shown as lines are the asymptotic

expressions (16) and (17) (dashed lines) and (20) (full lines). The plots and the asymptotic expressions show that $\langle G_m \rangle$ grows linearly with time t when the system is in the random deposition regime, the pre-factor being independent of the value of ν. After the system passes the first crossover point, a pre-factor that depends on ν shows up. In the correlated regime $\langle G_m \rangle$ approaches the steady-state value $G_{s,m}$ algebraically, with an exponent $\frac{d}{m}$.

We should point out that the plots of $\langle G_m \rangle$ versus time, shown in figure 2, do not easily reveal the crossovers between the different growth regimes, in contrast to the surface width where the different regimes are readily identified by mere inspection. This might somehow restrict the usefulness of $\langle G_m \rangle$, especially in situations where the steady state value is not readily known. Still, as we discuss in the following, the quantity G_m has many other advantages that makes it appealing both for theoretical and experimental studies.

3. Correlation function, response function, and fluctuation-dissipation ratio

Two-time quantities are extremely useful when studying relaxation phenomena, as they allow to capture most of the processes that underly the properties of a system far from its steady state. Typical quantities are the correlation and response functions, as well as combinations of these as for example the fluctuation-dissipation ratio. In the context of non-equilibrium growth processes and interface fluctuations both local [15, 16, 17, 18, 19, 20, 11] and global [21] quantities have been studied to some extent. The former include the height-height correlation function, whereas the change of the surface width to a global perturbation is an example of the latter. In the following we are deriving the corresponding exact expressions that involve the global quantity G_m.

We first consider the two-time correlation function of G_m. Starting from Eq. (11) we have

$$\begin{aligned}\langle G_m(t) G_m(s) \rangle = 2^{m-2} L^{2d} \sum_{\mathbf{q},\mathbf{p}\neq 0} \mathcal{P}(\mathbf{q})^{m/2} \mathcal{P}(\mathbf{p})^{m/2} \\ \Big\{ \Big\langle h_{\mathbf{q}}(t) h_{-\mathbf{q}}(t) \Big\rangle \Big\langle h_{\mathbf{p}}(s) h_{-\mathbf{p}}(s) \Big\rangle \\ + \Big\langle h_{\mathbf{q}}(t) h_{\mathbf{p}}(s) \Big\rangle \Big\langle h_{-\mathbf{q}}(t) h_{-\mathbf{p}}(s) \Big\rangle \\ + \Big\langle h_{\mathbf{q}}(t) h_{-\mathbf{p}}(s) \Big\rangle \Big\langle h_{-\mathbf{q}}(t) h_{\mathbf{p}}(s) \Big\rangle \Big\}. \end{aligned} \tag{21}$$

Here we note that the first term yields $\langle G_m(t) \rangle \langle G_m(s) \rangle$. Since $\mathcal{P}(\mathbf{q})$ is an even function, we have

$$\mathcal{P}(\mathbf{q})^{m/2} \mathcal{P}(-\mathbf{q})^{m/2} = \mathcal{P}(\mathbf{q})^{m/2} \mathcal{P}(\mathbf{q})^{m/2} = \mathcal{P}(\mathbf{q})^m. \tag{22}$$

Using the Eqs. (13) and (22) we obtain the connected two-point correlation function of the quantity G_m :

$$\begin{aligned} C_G(t,s) &\equiv \langle G_m(t) G_m(s) \rangle - \langle G_m(t) \rangle \langle G_m(s) \rangle \\ &= 2^{m-3} \frac{D^2}{\nu^2} \sum_{\mathbf{q}\neq 0} \frac{1}{q^{2m}} e^{-2q^m \nu t} \left(e^{2q^m \nu s} + e^{-2q^m \nu s} - 2 \right) \mathcal{P}(\mathbf{q})^m. \end{aligned} \tag{23}$$

We can also calculate the evolution of the average of G_m subjected to a perturbation when we suddenly change ν during the growth process. This is a natural thing to do as in the effective Hamiltonian (4) ν and G_m are conjugate quantities. Note that this is the response of a global quantity to a global perturbation.

In this case, the evolution of the average of G_m can be written as

$$\langle G_m \rangle_{\mu\to\nu}(t,s) = 2^{\frac{m-2}{2}} L^d \sum_{\mathbf{q}\neq 0} \left\langle h_{\mathbf{q}}^{(\mu\to\nu)} h_{-\mathbf{q}}^{(\mu\to\nu)} \right\rangle \mathcal{P}(\mathbf{q})^{m/2}, \tag{24}$$

where the notation "$\mu\to\nu$" indicates that the change from μ to ν at the waiting time s.

As discussed in [21, 20], the solution of the Langevin equation at times $t > s$ becomes under that change

$$h_{\mathbf{q}}^{(\mu\to\nu)}(t) = e^{-\nu\mathbf{q}^m(t-s)} h_{\mathbf{q},\mu}(s) + \int_s^t d\tau e^{-\nu\mathbf{q}^m(t-\tau)} \eta_{\mathbf{q}}(\tau), \tag{25}$$

where

$$h_{\mathbf{q},\mu}(s) = \int_0^s d\tau e^{-\mu\mathbf{q}^m(s-\tau)} \eta_{\mathbf{q}}(\tau) \tag{26}$$

is the solution of a surface that evolves until time s at the value μ when starting from a flat initial state.

Plugging Eq. (25) into Eq. (24) and comparing with Eq. (14), we straightforwardly obtain the response function

$$\begin{aligned}\chi_G(t,s) &\equiv \frac{\langle G_m\rangle_{\mu\to\nu}(t,s) - \langle G_m\rangle_{\nu}(t)}{\epsilon} \\ &= 2^{\frac{m-4}{2}} \frac{D}{\epsilon} \sum_{\mathbf{q}\neq 0} \frac{1}{q^m} e^{-2q^m\nu(t-s)} \left\{ \frac{1}{\mu}\left(1 - e^{-2q^m\mu s}\right) \right. \\ &\quad \left. -\frac{1}{\nu}\left(1 - e^{-2q^m\nu s}\right) \right\} \mathcal{P}(\mathbf{q})^{\frac{m}{2}},\end{aligned} \tag{27}$$

where $\epsilon = \nu - \mu$.

With these exact expressions for the correlation and response functions, we can now discuss the properties of the two-time quantities in the different regimes. In addition we can also study the behavior of composite quantities, as for example the fluctuation-dissipation ratio:

$$X(t,s) \equiv \frac{\partial\chi_G(t,s)}{\partial s} \Big/ \frac{\partial C_G(t,s)}{\partial s}. \tag{28}$$

The asymptotic steady-state behavior (i.e. the large νs limit) is easy to obtain for both quantities. Considering only the contribution from terms with minimum $\mathbf{q}$, we have

$$C_G(t,s) \approx \frac{dD^2}{4\nu^2} e^{-2q_{min}^m\nu(t-s)} \tag{29}$$

and

$$\chi_G(t,s) \approx \frac{dD}{2\nu^2} e^{-2q_{min}^m\nu(t-s)}, \tag{30}$$

where we have applied the approximation

$$\mathcal{P}(\mathbf{q}_{min})/q_{min}^2 \approx 1/2 \tag{31}$$

to both expressions and, in addition, the limit $\epsilon = \nu - \mu \to 0$ to the response function. For the fluctuation-dissipation ratio (28) we obtain in the steady state, with $t \longrightarrow \infty$,

$$X = \frac{2}{D}. \tag{32}$$

Assuming the validity of the Einstein relation $D = 2T$, which makes the steady state to be an equilibrium steady state, we recover the fluctuation-dissipation theorem $X = 1/T$, as expected.

The most interesting case is the case where both the waiting and observation times are in the correlated regime (to leading order identical results are obtained when the waiting time is still in the initial RD regime). This corresponds to $1/q_{max} < l_s, l_t < 1/q_{min}$, with $l_s < l_t$ and $l_t \gg 1$. In this limit, we replace in both Equations (23) and (27) the sums by integrals and treat the integrands as hyperspherical symmetric functions (this is the same method we used to derive the asymptotic equation for $\langle G_m \rangle$ in the correlated regime). We thereby obtain the following power-law decay functions

$$C_G(t,s) \approx D^2 s^2 \left(\frac{L}{2\pi}\right)^d \Omega_d \frac{\Gamma\left(2+\frac{d}{m}\right)}{2m} (2\nu t)^{-2-\frac{d}{m}}, \tag{33}$$

and

$$\chi_G(t,s) \approx D s^2 \left(\frac{L}{2\pi}\right)^d \Omega_d \frac{\Gamma\left(2+\frac{d}{m}\right)}{2m} \left[2\nu(t-s)\right]^{-2-\frac{d}{m}} \tag{34}$$

where we replaced $e^{2q^m\nu s} + e^{-2q^m\nu s} - 2$ by $(2q^m\nu s)^2$ for the correlation function and $(1 - e^{-2q^m\mu s})/\mu - (1 - e^{-2q^m\nu s})/\nu$ by $2(\nu - \mu)s^2q^{2m}$ for the response function before integrating. These replacements correspond to retaining only the leading terms in the Taylor expansions.

In the literature on physical ageing it is convention to write for a system undergoing simple ageing the two-time correlation and integrated response functions in the form [14]

$$C_G(t,s) = s^{-b} f_C(t,s) \quad ; \quad \chi_G(t,s) = s^{-a} f_\chi(t,s)\ , \tag{35}$$

where $f_C(y)$ and $f_\chi(y)$ are scaling functions which decay algebraically for large arguments:

$$f_C(y) \sim y^{-\lambda_C/z} \quad ; \quad f_\chi(y) \sim y^{-\lambda_\chi/z}\ , \tag{36}$$

where λ_C and λ_χ are the autocorrelation and autoresponse exponents, whereas z is the dynamical exponent (for our linear Langevin equations we have that $z = m$). Recasting Equations (33) and (34) in these ageing forms, we immediately obtain that $a = b = d/m$, whereas the autocorrelation and autoresponse exponents are given by $\lambda_C = \lambda_\chi = 2 + d/m$. This ageing scaling is illustrated in Figure 3 for the one-dimensional EW system.

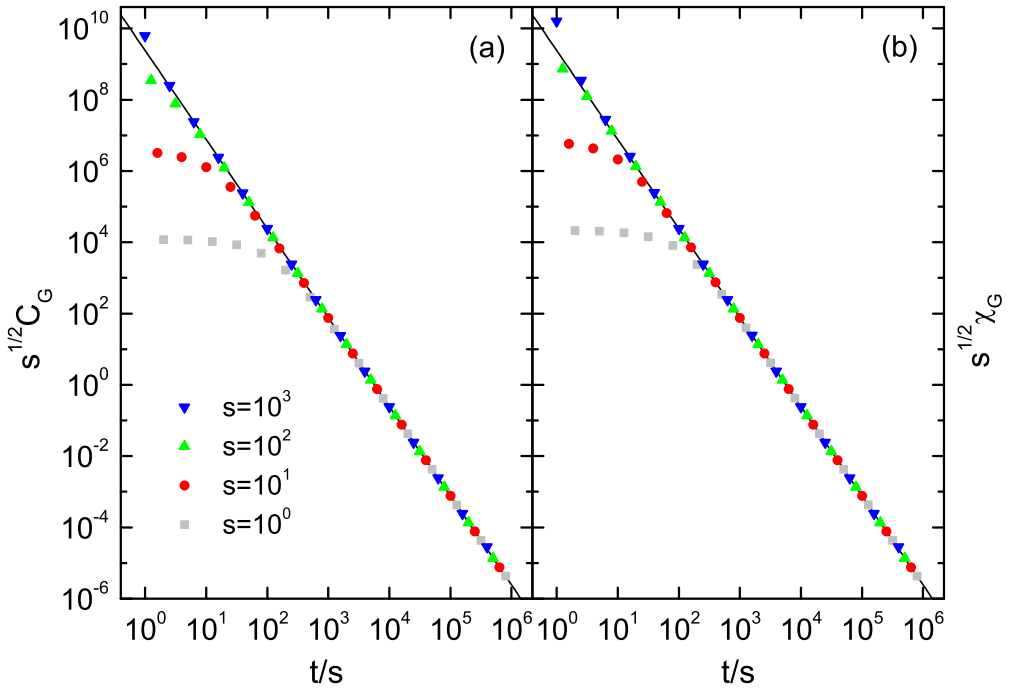

Figure 3. Ageing scaling of (a) the correlation and (b) the response function of G_2 in the one-dimensional EW system. The parameters for the calculations are $L = 2^{14}$, $\nu = 0.001$, and $D = 1$.

Let us close this Section with a more careful discussion of the fluctuation-dissipation ratio (28). As mentioned in the introduction, in a finite system the correlated regime goes over into the stationary state at a system size dependent time t_2. In the infinite system t_2 diverges and the correlated regime prevails for all times. For the limit value

$$X_\infty = \lim_{s\longrightarrow\infty} \lim_{t\longrightarrow\infty} X(t,s) \tag{37}$$

we therefore obtain the values $X_\infty = 1/D = 1/2T$ for the infinite system and $X_\infty = 2/D = 1/T$ for the finite system, where we introduced temperature via the Einstein relation. The finite system ending up in the steady state for finite times, we recover the fluctuation-dissipation ratio. If the system remains in the correlated region, the effective temperature is twice that of the heat-bath. The crossover between these two regimes can be visualized for finite systems by plotting $X(s)$ for $t \gg s$ as done in Fig. 4. For that figure we plot the value of $X(s) = X(s + 10^6, s)$ as a function of s, which yields the value $X(s) = 1/D$ for $s \ll t_2$ and the value $X(s) = 2/D$ for $s \gg t_2$. The crossover times for $m = 2$ and $m = 4$ are indicated in Fig. 4 by the vertical lines. Fig. 5 gives a more comprehensive view of the behaviour of $X(t, s)$ as a function of both s and t for the one-dimensional EW equation. Two plateaus can be distinguished in the contour plot: one for the steady state (i.e. the regime where $t > t_2$ and $s > t_2$) where $X = 2/D$ and one away from stationarity, with $t \gg 1/2\nu$ and $s < t_2$, where $X = 1/D$.

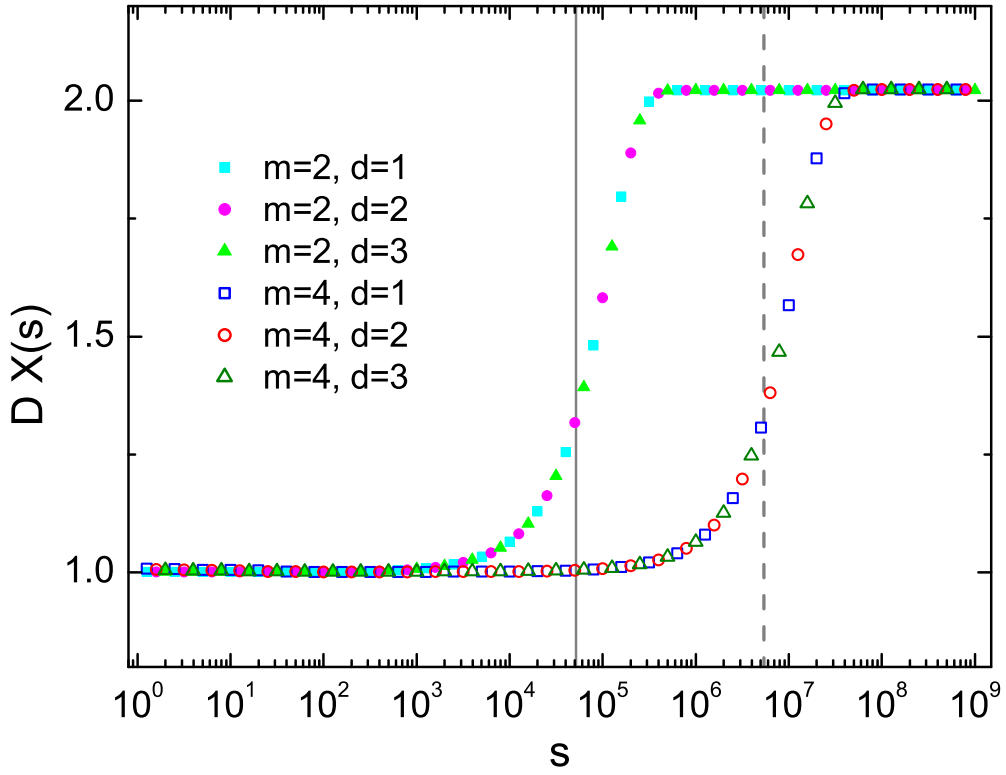

Figure 4. The fluctuation-dissipation ratio for $m = 2$ and $m = 4$ in various dimensions as a function of the waiting time s. Here $X(s) = X(s + 10^6, s)$. The linear extension for all systems is $L = 2^6$. The full (dashed) line indicates the crossover time t_2 for the EW (MH) equation. Note that the value of $X(s)$ is independent of the dimensionality of the substrate.

4. Conclusion

In the past the study of global quantities in systems relaxing towards a steady state has proven very fruitful in a large variety of systems (see the corresponding discussion in [14]). In [21] we did a first attempt at using global quantities in the context of correlated growth and interface fluctuations, choosing the surface width as our global quantity. However, the surface width is a complicated quantity that has the notable drawback that the conjugate system parameter is unknown. Consequently, it is not possible to form a meaningful fluctuation-dissipation ratio using that quantity.

In this paper we are proposing a different global quantity for the study of kinetic roughening and related interface problems. This quantity is proportional to the effective Hamiltonian used in the Langevin description and is conjugate to a system parameter that can be changed in experiments [20, 11]. In fact, G_m seems better suited to capture the roughness of a surface than the surface width itself, as illustrated in figure 1.

Focusing on linear Langevin equations we derive exact expressions for G_m as well as for the corresponding correlation and response functions. This allows us to discuss also more complicated quantities as for example the fluctuation-dissipation ratio. In fact, we recover for the quantities derived from G_m the fluctuation-dissipation theorem for

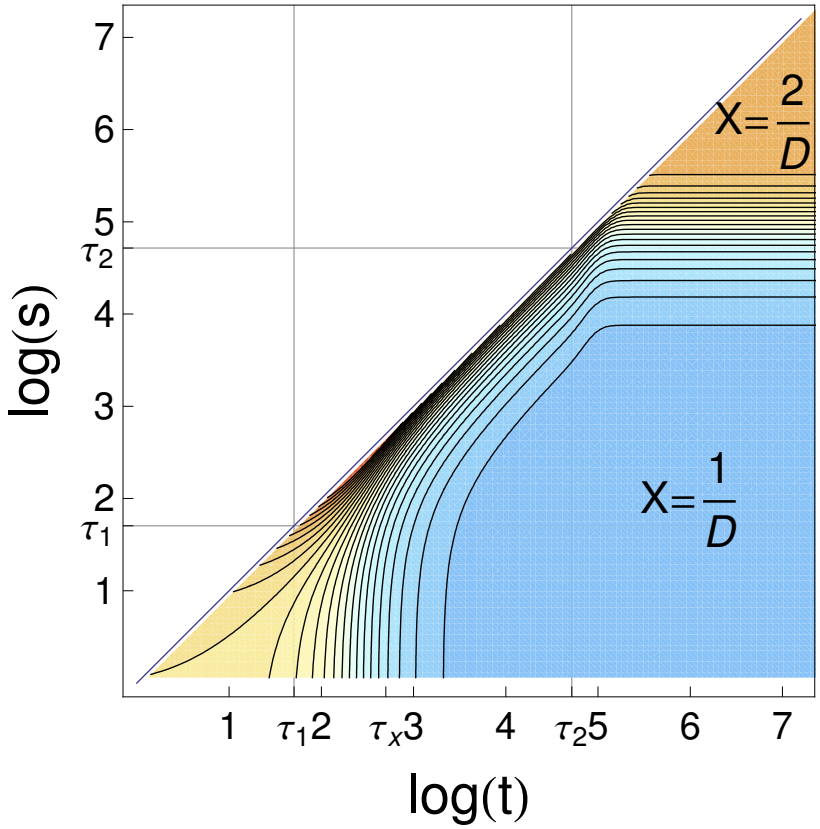

Figure 5. Contour plot of $X(t,s)$ for the one-dimensional EW equation. The parameters used here are $L = 2^6$, $\nu = 0.001$, and $\mu = 0.99\nu$. Also indicated are the logarithms of various relevant time scales: $\tau_1 = \log t_1$, $\tau_2 = \log t_2$, and $\tau_x = \log(1/2\nu)$. Here and in the following figures, we illustrate our results for the one-dimensional EW equation, but, as the exact results reveal, similar results are obtained for the MH equation as well as for substrates of higher dimensionality.

equilibrium steady states, whereas in the correlated regime we can assign an effective temperature to our system.

All calculations presented in this paper have been done in the context of linear Langevin equations. However, most growth processes are governed by non-linearities. It is therefore important to clarify to what extend our results obtained for linear Langevin equations remain valid when considering non-linear stochastic equations as for example the Kardar-Parisi-Zhang equations [27]. We intend to address this and other questions in the future.

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5. References

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